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EXPONENTIAL FOURIER DENSITIES ON SO(3) AND OPTIMAL ESTIMATION A--ETC(U)
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Exponential Fourier Densities on SO(3)
and
Optimal Estimation and Detection for Rotational Processes

by

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ABSTRACT

In this paper, we will present a new representation of a probability density function on the three dimensional rotation group, SO(3), which generalizes the exponential Fourier densities on the circle. As in the circle case, this class of densities on SO(3) is also closed under the operation of taking conditional distributions. Several simple multistage estimation and detection models will be considered in this paper. The closure property enables us to determine the sequential conditional densities by recursively updating a finite and fixed number of coefficients. It also enables us to express the likelihood ratio for signal detection explicitly in terms of the conditional densities.

An error criterion, which is compatible with a Riemannian metric, will be introduced and discussed in this paper. The optimal orientation estimates with respect to this error criterion will be derived for a given probability distribution, illustrating how the updated conditional densities can be used to sequentially determine the optimal estimates on SO(3).

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I. Introduction

Rigid body rotations are involved in many important practical problems of detection, estimation, and control. Some notable examples can be found in gyroscopic analysis and satellite attitude determination and control. While linearization and approximation techniques have led to many useful results, simple analytic tools which will enable us to analyze and synthesize the optimal structures have long been desired.

Optimal estimation and detection schemes for discrete-time processes whose state space is a circle or sphere have been obtained in [1] and [2] by using a novel representation for probability densities which has the form $\exp f$ where f is a finite linear combination of functions which form a complete orthogonal system on the state space involved. In the case of the circle, circular functions were used, while both spherical harmonics and multiple trigonometric functions were employed for densities defined on the sphere.

In this paper the same approach will be taken for discrete-time rotational processes by introducing a similar exponential density referred to as a rotational exponential Fourier density (REFD) defined on the group of rotations of three-dimensional space, that is obtained by using a sequence of irreducible unitary representations which form a complete orthogonal system on SO(3). It can be shown that a continuous density function on SO(3) can be approximated by such a REFD as closely as we wish in the space of square integrable functions.

As in the circle case, the class of REFD's of a certain finite order is closed under the operation of taking conditional distributions as a consequence of the group structure of SO(3). We note that there does not exist such a closure property in the sphere case [2] and a combined usage of two kinds of exponential densities is required to treat analogous estimation and detection problems on the sphere. It will become clear in the sequel that it is exactly this closure property of REFD's that yields simple estimation and detection schemes which update the sequential conditional densities by recursively revising a finite and fixed number of parameters.

One drawback of REFD's is that there is no known closure property of convolution, which places a restriction on the type of rotational signal processes that can be considered for the above approach. More specifically, the rotational signal process considered in this paper disallows random driving terms. Given two orientations of a rigid body which are represented by two points on SO(3), the minimal angle in radians required to bring one into the other is a Riemannian metric on SO(3) and is a natural measure of the distance between them.

An error criterion for orientation estimation, which is compatible with this measure of distance will be introduced in this paper. It is compatible in the sense that it is an increasing function of this Riemannian metric. Various descriptions and properties of this criterion will be discussed. The optimal orientation estimate and its estimation error with respect to this error criterion will be derived for a given probability distribution, thereby illustrating how the updated conditional densities can be used to sequentially determine the optimal estimates of the rigid body orientations.

It should be mentioned that estimation for continuous-time directional processes on SO(3) has been considered in [9] and [10].

II. Preliminaries and Rotational Exponential Fourier Densities

A rotation of three-dimensional Euclidean space about a fixed point can be described analytically as a linear transformation of the space that preserves distances, leaving the origin unchanged, which can be represented by an orthogonal matrix. In this paper we will be concerned with the group of rotations, denoted by SO(3), that consists of those rotations whose matrix representation has determinant equal to +1; that is, we are excluding those rotations that consist of a rotation followed by a reflection. There are several different ways of parametrizing SO(3) such as by Euler angles, unit quaternions, direction cosines, and Cayley-Klein parameters which are all discussed in [3]. While we will have occasion to refer to all of these methods, it will be customary to parametrize any rotation R belonging to SO(3) by a triple of Euler angles (ϕ,θ,ψ) which determine the rotation according to the following sequence of rotations [4, p.107]:

(i) a rotation through the angle $\phi, 0 \leq \phi < 2\pi$, about the z-axis,

(ii) a rotation through the angle $\theta, 0 \leq \theta < \pi$, about the new x-axis,

(iii) a rotation through the angle $\psi, 0 \leq \psi < 2\pi$, about the new z-axis.

These rotations are illustrated in Figure 1, with $\bar{x}$, $\bar{y}$, $\bar{z}$ being the final position of the three coordinate axes.

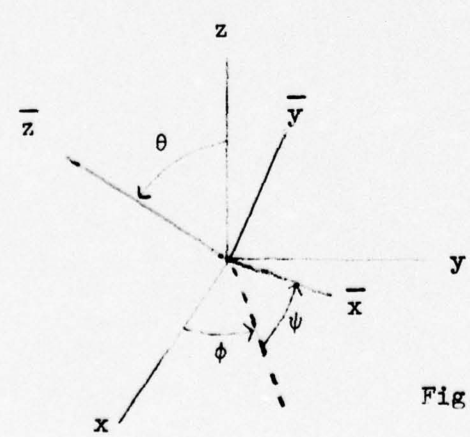

Fig. 1

An orthogonal matrix R that corresponds to this rotation can be obtained as the product of the three matrices corresponding to the rotations (i), (ii), (iii): $R = Z(\phi)X(\theta)Z(\psi)$ where

$$Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

so that

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\cos\phi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{bmatrix}$$

We note that each element of SO(3) can be parametrized by some triple of Euler angles; however, when $\theta = 0$ or π this parametrization is not unique.

To obtain useful special functions that are defined on SO(3) we shall consider the matrix elements of a sequence of group representations of SO(3). It should be recalled that a group representation, which is described in detail in [5] - [7] can be thought of as simply a group of matrices to which the group SO(3) is homomorphic. In particular, we will use the sequence of finite-dimensional unitary representations $\{D^{\ell}(\phi,\theta,\psi), \ell = 0,1,\ldots\}$ attributed to E. P. Wigner whose components are described in [6, p.144] by:

(2) $$D^{\ell}(\phi,\theta,\psi)_{mn} = i^{m-n}e^{-im\phi}d^{\ell}_{mn}(\theta)e^{-in\psi}$$

$$(3)\quad d^{\ell}_{mn}(\theta) = \frac{\sin^{n-m}\theta(1+\cos\theta)^m}{2^{\ell}[(\ell+m)!(\ell-m)!]^{1/2}}\left[\frac{(\ell-n)!}{(\ell+n)!}\right]^{1/2}\frac{d^{\ell+n}}{d(\cos\theta)^{\ell+n}}(\cos\theta-1)^{\ell+m}(1+\cos\theta)^{\ell-m},$$

where m and n are integers such that $-\ell \leq m,n \leq \ell$.

It is proved in [7, pp.233-284] that the functions $D^{\ell}(R)_{mn} \triangleq D^{\ell}(\phi,\theta,\psi)_{mn}$ form a complete orthogonal system in the space of square integrable functions $f(\phi,\theta,\psi)$, $0 \leq \theta < \pi$, $0 < \phi,\psi < 2\pi$, with respect to the inner product

$$\langle f_1,f_2\rangle = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f_1(\phi,\theta,\psi)\, f_2(\phi,\theta,\psi)\, \sin\theta d\phi d\theta d\psi .$$

The completeness of this system in the Hilbert space of square integrable functions defined on the rotation group is analogous to that of the circular functions on the circle, S^1, and, also, that of the spherical harmonics on the sphere, S^2.

We now define a <u>rotational exponential Fourier density of order N</u> to be denoted by REFD(N) as a probability density defined on SO(3) of the form

$$(4)\qquad p(R) = p(\phi,\theta,\psi) = \exp f(\phi,\theta,\psi)$$

$$f(\phi,\theta,\psi) = \sum_{\ell=0}^{N}\sum_{m,n=-\ell}^{\ell} a^{\ell}_{mn} D^{\ell}(\phi,\theta,\psi)_{mn}$$

where a^0_{00} is a normalizing constant and all other coefficients a^{ℓ}_{mn} are arbitrary complex numbers. In addition to the completeness property, another reason for our choosing this class of special functions is that for continuous densities we have the following approximation theorem, which is a generalization of Theorem 1 in [1].

Theorem 1. Let p be a probability density on SO(3). Assume that p is continuous. Then for any given positive number ε, there exists an REFD, $p_N(\phi,\theta,\psi) := \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a^{\ell}_{mn} D^{\ell}(\phi,\theta,\psi)_{mn}$, such that $||p - p_N|| \leq \varepsilon$.

Proof. Assume that

$$\inf\{p(x) : x \in SO(3)\} = c > 0 .$$

This assumption can be removed in exactly the same way as in the proof of Theorem 1 of [1]. We note that $f(x) = \ell n\, p(x)$ is now well defined and also continuous on SO(3).

Since SO(3) is compact, in view of the Peter-Weyl Theorem [6, p.99], for any $0 < \delta < 1$, there is a linear conbination of D^{i}_{mn}, say $f_\delta = \sum_{\ell=0}^{N(\delta)} \sum_{m,n=-\ell}^{\ell} a^{\ell}_{mn}(\delta) D^{\ell}_{mn}$, such that $||f_\delta - f||_\infty < \varepsilon$. It follows that $||f_\delta||_\infty < 1 + ||f||_\infty = :M$.

Define a function $g : R^1 \to R^1$ by

$$g(x) = \begin{Bmatrix} \exp x , & x \leq M \\ \exp M , & x \geq M \end{Bmatrix}$$

and an operator G on the set of real functions on SO(3) by $(Gu)(x) = g(u(x))$. It is obvious that g satisfies the Caretheodory conditions [14, p.20] and G transforms every function in $L^2(SO(3))$ into a function in $L^2(SO(3))$.

By Theorem 2.1 of [14, p.22], the operator G is continuous. Hence given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $||f_1 - f|| < \delta$, then $||Gf_1 - Gf|| < \varepsilon$. We choose $f_1 = f_\delta$. Then $||\exp f_\delta - p|| = ||Gf_\delta - Gf|| < \varepsilon$. This completes the proof.

Remark. We note that f_δ is not necessarily the $N(\delta)$-th order partial sum $s_{N(\delta)}f$ of the Fourier expansion of f. Thus in contrast to Theorem 1 of [1], it is not necessarily true that $\lim_{k\to\infty}||\exp s_k f - p|| = 0$, which is very useful in determining p_N in the statement of Theorem 1. Such a deficiency can be removed, if we require p to be twice continuously differentiable. This extra requirement ensures that $\ln p$ for $p > 0$, may be expanded in an absolutely and uniformly convergent Fourier series in terms of the rotational harmonics, D^i_{mn} [15, p.513]. Thus the Peter-Weyl Theorem is not needed for uniform approximation of a continuous function.

A remark is in order about (4). From our choice of Euler angle parametrization, it is easily seen that if the range of permissible values of ϕ,θ,ψ is ignored, then a rotation R^1 with Euler angles $(\phi+\pi, -\theta,\psi+\pi)$ would be equivalent to the rotation R with Euler angles (ϕ,θ,ψ) in the sense that the final positions of the coordinate axes would be identical. Consequently, it is advantageous to extend the definition of the function $D^\ell(\phi,\theta,\psi)_{mn}$ so that we have the property

$$p(R) = p(R^1) \tag{5}$$

and so that we can lift our restrictions on θ, ϕ, and ψ, except $\theta \neq 0$. First we extend the definition of $d^\ell_{mn}(\theta)$ to include all values of θ such that $0 < |\theta| < \pi$ by defining

$$d^\ell_{mn}(\theta) = (-1)^{m+n} d^\ell_{mn}(-\theta) , \quad -\pi < \theta < 0 .$$

Using (2) we see that (5) holds. For the situation where $\theta = 0$, it is shown in [6, p.114] that

$$D^\ell(\phi,0,\psi)_{mn} = e^{-im(\phi+\psi)}\delta_{mn} ;$$

thus, since $\phi + \psi$ is the fixed amount of rotation about the z-axis even though each angle is not uniquely determined, we are assured that for any choice of Euler angles representing this rotation as well as any other rotation R the density (4) is well-defined.

Before proceeding to consider an estimation problem on SO(3), we enumerate some properties of the functions $D^{\ell}(R)_{mn}$ and $d^{\ell}_{mn}(\theta)$ which will be employed in Sections IV and V:

(6) $$D^{\ell}(R_1R_2)_{mn} = \sum_{s=-\ell}^{\ell} D^{\ell}(R_1)_{ms}D^{\ell}(R_2)_{sn}$$

for any rotations R_1 and R_2 belonging to SO(3), [7, p.281];

(7) $$d^{\ell}_{mn}(\theta) = (-1)^{n-m}d^{\ell}_{-m,-n}(\theta) = d^{\ell}_{-n,-m}(\theta)\ , \quad [6, p.157];$$

(8) $$D^{\ell}(R)_{mn}D^{\ell'}(R)_{m'n'} = e^{im\phi}\sum_{L} (2L+1) \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix}\begin{pmatrix} \ell & \ell' & L \\ n & n' & N \end{pmatrix} d^{L}_{MN}(\theta)e^{in\psi}$$

where $M = -m - m'$ and $N = -n - n'$, [6, p.157]. The summation is over all L for which $|\ell-\ell'| \leq L \leq \ell + \ell'$ and for which the 3-j coefficients of Wigner,

$$\begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ell & \ell' & L \\ n & n' & N \end{pmatrix} ,$$

exist; these coefficients are defined in [6, p.120] by

$$\begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix} = (-1)^{2j-k+n} \left[\frac{(j+k-\ell)!(k+\ell-j)!(\ell+j-k)!(\ell+p)!(\ell-p)!}{(j+k+\ell+1)!(j+m)!(j-m)!(k+n)!(k-n)!}\right]^{1/2}$$

$$\times \sum_{t} (-1)^{t} \frac{(\ell+j-n-t)!(k+n+t)!}{(\ell+p-t)!(t+k-j-p)!t!(\ell-k+j-t)!}$$

where the summation is over all integers such that the arguments of all factorials are nonnegative.

III. An Error Criterion and Optimal Estimates on SO(3).

In order to define an error criterion for orientation estimation, it is necessary to have a measure of the distance between two orientations. We will first describe such a measure, using quaternions [11]. We recall that a rotation about an axis in the direction of a unit vector $[\ell,m,n]'$ through an angle Φ is represented by the (unit) quaternion

$$q = [q_1,q_2,q_3,q_4]' = [\cos\frac{\Phi}{2}, \ell\sin\frac{\Phi}{2}, m\sin\frac{\Phi}{2}, n\sin\frac{\Phi}{2}]'$$

Given two orientations, the minimal angle in radians required to bring one into the other is a natural measure of distance between them and defines a Riemannian metric on SO(3). If the orientations are represented by the quaternions, q and p, and the minimal angle is denoted by $\rho(q,p)$, then we have $q'p = \cos\frac{1}{2}\rho(q,p)$. As $\frac{1}{2}(1 - \cos\rho)$ is a monotone increasing function of ρ, a measure of distance between p and q can be defined to be $\frac{1}{2}(1 - \cos\rho(q,p)) = 1 - (q'p)^2$. It can be shown that if the orientations, q and p, are described by the 3x3-dimensional orthogonal matrices, Q and P, then this measure of distance can also be expressed as $\frac{1}{4}(3 - \mathrm{tr}\, PQ')$.

We are now ready to define the error criterion for orientation estimation. Let q be a random quaternion and p its estimate. Then a measure of the estimation error is

$$J(q,p) = E(1 - (q'p)^2) . \tag{9}$$

If the probability distribution of q is given, the estimate p which minimizes J may be obtained from observing that

$$J(q,p) = 1 - p' E(qq')p .$$

It is well-known [12, p.142] that the quadratic form $p'Vp$ of the positive definite matrix $V = E(qq')$ is maximized when p is the unit eigenvector associated with the largest eigenvalue λ of V. Moreover, the maximum value is λ. Hence,

$$\begin{aligned} \min_p J(q,p) &= 1 - \hat{q}'E(qq')\hat{q} \\ &= 1 - \lambda \end{aligned}$$

where

λ = the maximum eigenvalue of $E(qq')$

$\hat{q}$ = the unit eigenvector of $E(qq')$ associated with λ.

The probability distributions on SO(3) are expressed in terms of the Euler angles (ϕ,θ,ψ) in this paper. The following relationships [10, p.380] between the quaternion components and the Euler angles will have to be used to calculate the optimal estimate $\hat{q}$ and its estimation error $1 - \lambda$:

$$\begin{aligned} q_1 &= \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2} \\ q_2 &= \sin\frac{\theta}{2}\cos\frac{\phi-\psi}{2} \\ q_3 &= \sin\frac{\theta}{2}\sin\frac{\phi-\psi}{2} \\ q_4 &= \cos\frac{\theta}{2}\sin\frac{\phi+\psi}{2} \end{aligned} \tag{10}$$

Once vector $\hat{q}$ is determined we can immediately determine the Euler angles $(\hat{\phi},\hat{\theta},\hat{\psi})$ for the optimal orientation estimate from the relations:

$$
(11) \qquad \begin{aligned}
&\cos\hat{\theta} = 2(\hat{q}_1^{\,2} + \hat{q}_4^{\,2}) - 1 , && 0 \leq \hat{\theta} \leq \pi \\
&\sin\hat{\phi} = -\frac{1}{\Delta}(\hat{q}_3\hat{q}_1 + \hat{q}_2\hat{q}_4) , && \cos\hat{\phi} = \frac{1}{\Delta}(\hat{q}_1\hat{q}_2 - \hat{q}_3\hat{q}_4) \\
&\sin\hat{\psi} = \frac{1}{\Delta}(\hat{q}_2\hat{q}_4 - \hat{q}_1\hat{q}_3) , && \cos\hat{\psi} = \frac{1}{\Delta}(\hat{q}_1\hat{q}_2 + \hat{q}_3\hat{q}_4) \\
&\Delta = \sqrt{(\hat{q}_1^{\,2}+\hat{q}_4^{\,2})(\hat{q}_2^{\,2}+\hat{q}_3^{\,2})}
\end{aligned}
$$

which are the inverse of the above set of relationships (10).

IV. Estimation for Rotational Processes with Rotational Noise.

Suppose s is a rotation of a rigid body with Euler angles (ϕ,θ,ψ) and v is a second rotation with Euler angles $(\varepsilon,\eta,\delta)$. We can obtain a product rotation m with Euler angles $(\tilde{\phi},\tilde{\theta},\tilde{\psi})$ by the successive rotation of the rigid body by s and then v. We denote this product by

$$
(12) \qquad m(\tilde{\phi},\tilde{\theta},\tilde{\psi}) = v(\varepsilon,\eta,\delta) \cdot s(\phi,\theta,\psi) .
$$

Our reason for writing v to the left of s is to be consistent with the matrix equation resulting by the use of the orthogonal matrix representation where the matrix S representing s is pre-multiplied by the matrix V representing v to obtain $M = VS$.

Performing the indicated multiplication when each matrix has the form (1) and equating respective elements, we may obtain nine relationships among the Euler angles of m, v, and s, which uniquely determine $(\tilde{\phi},\tilde{\theta},\tilde{\psi})$ in terms of $(\varepsilon,\eta,\delta)$ and (ϕ,θ,ψ). The relationships are very cumbersome and thus will not be displayed here.

We now consider a rotational signal process which is a sequence of rotations $\{s_k\}_{k=0}^{\infty}$ defined on $SO(3)$ whose terms are related by the equation

(13) $$s_{k+1} = w_k \circ s_k$$

where w_k is a deterministic rotation whose Euler angles are known. Let $\{m_k\}_{k=1}^{\infty}$ and $\{v_k\}_{k=1}^{\infty}$ be sequences of rotation on SO(3) that constitute a measurement process and a measurement noise process, respectively, such that

(14) $$m_k = v_k \circ s_k$$

The first estimation problem to be solved can now be stated as being that of finding the optimal estimate of s_k given the set of measurements

$$m^k \triangleq \{m_1,\ldots,m_k\} \ , \ k = 1,2,\ldots$$

using the error criterion (9). From our results in Section III we recognize the fact that in order to determine the optimal estimate which involves conditional expectations, it is crucial to be able to produce the conditional density $p(s_k|m^k)$, which we note can be calculated, by Bayes rule from

(15) $$p(s_k|m^k) = C_k p(m_k|s_k)p(s_k|m^{k-1})$$

where C_k is a normalizing constant.

It will now be demonstrated that REFD's introduced in Section II are ideal ones to use in computing this conditional density. Suppose s_0 and $\{v_k\}$ have REFD(N) (if they have different orders, we can let N be the maximum order and by inserting zero coefficients make all densities of order N) which are described, respectively, by

(16) $$p(s_0) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell 0} D^{\ell}(s_0)_{mn}$$

(17) $$p(v_k) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell k} D^{\ell}(v_k)_{mn}$$

Let us further assume that the conditional density $p(s_{k-1}|m^{k-1})$ is a REFD(N), denoted by

$$p(s_{k-1}|m^{k-1}) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell,k-1} D^{\ell}(s_{k-1})_{mn} . \tag{18}$$

Under these assumptions we will show that $p(s_k|m^k)$ is also a REFD(N) and at the same time exhibit a recursive formula for the Fourier coefficients $a_{mn}^{\ell k}$.

From (13) and (14) and the group property of SO(3), v_k and s_{k-1} can be expressed as

$$v_k = m_k \circ s_k^{-1} \quad \text{and} \quad s_{k-1} = w_{k-1}^{-1} \circ s_k$$

so that, using (18), $p(s_k|m^{k-1})$ is a REFD:

$$\begin{aligned} p(s_k|m^{k-1}) &= \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell,k-1} D^{\ell}(w_{k-1}^{-1} \circ s_k)_{mn} \\ &= \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell,k-1} \sum_{s=-\ell}^{\ell} D^{\ell}(w_{k-1}^{-1})_{ms} D^{\ell}(s_k)_{sn} \\ &= \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} \left[\sum_{j=-\ell}^{\ell} a_{jn}^{\ell,k-1} D^{\ell}(w_{k-1}^{-1})_{jm} \right] D^{\ell}(s_k)_{mn} ; \end{aligned} \tag{19}$$

for the second equality, property (6) was used while the third equality is obtained by a change in summation indices.

Likewise, the conditional density $p(m_k|s_k)$ is a REFD:

$$\begin{aligned} p(m_k|s_k) &= \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell k} D^{\ell}(m_k \circ s_k^{-1})_{mn} \\ &= \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} \left[\sum_{j=-\ell}^{\ell} b_{jn}^{\ell k} D^{\ell}(m_k)_{jm} \right] D^{\ell}(s_k^{-1})_{mn} \\ &= \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} \left[(-1)^{m+n} \sum_{j=-\ell}^{\ell} b_{j,-m}^{\ell k} D^{\ell}(m_k)_{j,-n} \right] D^{\ell}(s_k)_{mn} , \end{aligned} \tag{20}$$

the last equality results from a change of summation indices and the fact that if s_k has Euler angles (ϕ,θ,ψ) then s_k^{-1} has Euler angles $(\pi-\psi,\theta,\pi-\phi)$ so that, using (2) and (7) ,

$$\begin{aligned} D^{\ell}(s_k^{-1})_{mn} &= i^{m-n}(-1)^{m+n}e^{im\psi}d^{\ell}_{mn}(\theta)e^{in\phi} \\ &= (-1)^{m+n}i^{-n+m}e^{im\psi}d^{\ell}_{-n,-m}(\theta)e^{in\phi} \\ &= (-1)^{m+n}D^{\ell}(s_k)_{-n,-m} \end{aligned}$$

Substituting (19) and (20) into (15), we obtain the REFD

$$\begin{aligned} p(s_k|m^k) = C_k \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} \Bigg[\sum_{j=-\ell}^{\ell} \Big\{ a_{jn}^{\ell,k-1}\, D^{\ell}(w_{k-1}^{-1})_{jm} \\ + (-1)^{m+n}\, b_{j,-m}^{\ell k}\, D^{\ell}(m_k)_{j,-n} \Big\} \Bigg] D^{\ell}(s_k)_{mn} \end{aligned}$$

Thus $p(s_k|m^k)$ is a REFD(N) whenever $p(s_{k-1}|m^{k-1})$ is a REFD(N) but since

$$p(s_0|m_0) \triangleq p(s_0)$$

is itself a REFD(N) by induction we have proved the following:

<u>Theorem 2.</u> If $\{s_k\}$ is a sequence of random rotations on SO(3) where

$$s_{k+1} = w_k \circ s_k$$

for some sequence of deterministic rotations and m_k is the corresponding measurement sequence where each s_k has been corrupted by a noise rotation v_k such that

$$m_k = v_k \circ s_k$$

then if s_0 and $\{v_k\}$ have REFD's given by (16) and (17), respectively, then, for $k = 1,2,\ldots,$ the conditional density $p(s_k|m^k)$ is a REFD(N) of the form

$$p(s_k|m^k) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell k} D^{\ell}(s_k)_{mn}$$

where the coefficients $a_{mn}^{\ell k}$ are determined recursively by the formulas

$$a_{mn}^{\ell k} = \sum_{j=-\ell}^{\ell} \{a_{jn}^{\ell,k-1} D^{\ell}(w_{k-1}^{-1})_{jm} + (-1)^{m+n} b_{j,-m}^{\ell k} D^{\ell}(m_k)_{j,-n}\} \text{ , } \ell \neq 0,\ k = 1,2,3,\ldots;$$

and a_{00}^{0k} is a normalizing constant.

This theorem enables us to calculate the sequential conditional densities by updating recursively a finite and fixed number of parameters. Using the conditional density $p(s_k|m^k)$ at time k , the optimal estimate of the orientation can be determined as suggested at the end of Section III. Namely we first compute the conditional covariance matrix $E(q(k)q'(k)|m^k)$ where $q(k)$ is the quaternion for s_k whose components expressed in terms of the Euler angles are given by (10). Then we use some standard numerical method such as the power method [13, pp.147-150] to compute the largest eigenvalue $\lambda(k)$ and the associated unit eigenvector $\hat{q}(k|k)$. The Euler angles of the optimal estimate may then be determined from $\hat{q}(k|k)$ through (11). The estimation error is $1 - \lambda(k)$.

V. Estimation for Rotational Processes with Additive White Gaussian Noise.

A second model for which the estimation problem can be solved using REFD's is described by the equations

(21) $$s_{k+1} = w_k \circ s_k$$

(22) $$m_k = h(s_k) + v_k$$

where $\{s_k\}$ is again a sequence of random rotations belonging to SO(3) and $\{w_k\}$ is now a sequence of p-dimensional vectors of observed outputs of the signal process, h is a vector-valued function defined on SO(3) that is square-integrable, and $\{v_k\}$ is a sequence of p-dimensional independent Gaussian vectors, each with zero mean value and covariance matrix

$$R_k = E[v_k v_k']$$

The completeness property of the functions $\{D^\ell_{mn}\}$ assures us that, for any $\varepsilon > 0$ and for each component h^j of function h, there exists an integer M_j and coefficients $\tilde{h}^{\ell j}_{mn}$ such that

$$||h^j(s) - \sum_{\ell=0}^{M_j} \sum_{m,n=-\ell}^{\ell} \tilde{h}^{\ell j}_{mn} D^\ell(s)_{mn}||_2 < \varepsilon \ , \ j = 1,2,\ldots,p \ .$$

Let $M = \max_j M_j$ and denote by $h_M(s)$ the p-dimensional vector whose components are $h^j_M(s)$ where

$$h^j_M(s) = \sum_{\ell=0}^{M} \sum_{m,n=-\ell}^{\ell} h^{\ell j}_{mn} D^\ell(s)_{mn}$$

$$h^{\ell j}_{mn} = \begin{cases} \tilde{h}^{\ell j}_{mn}, & \ell = 0,\ldots,M_j; \ -\ell \leq m,n \leq \ell \\ 0 \ , & \text{otherwise} \end{cases}$$

Each noise vector has density

$$p(v_k) = (2\pi)^{-p/2}(\det R_k)^{-1/2} \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{p} R_k^{ij} v_k^i v_k^j\right\}$$

where R_k has components R_k^{ij} and v_k has components v_k^i so if we replace (22) by

$$m_k = h_M(s_k) + v_k$$

then

$$p(m_k|s_k) = (2\pi)^{-p/2}(\det R_k)^{-1/2}\exp\left\{\frac{-1}{2}\sum_{i,j=1}^{p} R_k^{ij}\left[m_k^i - \sum_{\ell=0}^{M}\sum_{m,n=-\ell}^{\ell} h_{mn}^{\ell i} D^{\ell}(s_k)_{mn}\right]\right.$$

$$\left.\times\left[m_k^j - \sum_{\ell=0}^{M}\sum_{m,n=-\ell}^{\ell} h_{mn}^{\ell j} D^{\ell}(s_k)_{mn}\right]\right\}$$

(23)

$$= (2\pi)^{-p/2}(\det R_k)^{-1/2}\exp\left\{C_0 + \sum_{\ell=0}^{M}\sum_{m,n=-\ell}^{\ell} C_{mn}^{\ell} D^{\ell}(s_k)_{mn}\right.$$

$$\left.+ \sum_{\ell,\ell'=0}^{M}\sum_{m,n=-\ell}^{\ell}\sum_{m',n'=-\ell'}^{\ell'} C^{\ell\ell'}(m,n,m',n')D^{\ell}(s_k)_{mn}D^{\ell}(s_k)_{m'n'}\right\}$$

where

$$C_0 = -\frac{1}{2}\sum_{j=1}^{p} R_k^{ij}\, m_k^i\, m_k^j$$

$$C_{mn}^{\ell} = \frac{1}{2}\sum_{i,j=1}^{p} R_k^{ij}\left[m_k^j\, h_{mn}^{\ell j} + m_k^i\, h_{mn}^{\ell i}\right]$$

$$C^{\ell\ell'}(m,n,m'n') = -\frac{1}{2}\sum_{i,j=1}^{p} R_k^{ij} h_{mn}^{\ell i}\, h_{m'n'}^{\ell' j}\ .$$

But properties (7) and (8) imply that

$$\text{(24)}\qquad D^{\ell}(s_k)_{mn}D^{\ell'}(s_k)_{m'n'} = \sum_{L=|\ell-\ell'|}^{\ell+\ell'} (2L+1)(-1)^{N-M}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\begin{pmatrix}\ell & \ell' & L\\ n & n' & N\end{pmatrix} D^{L}(s_k)_{-M,-N}$$

so that the expresion given in (23) is a REFD(2M). Now, if $p(s_{k-1}|m^{k-1})$ is a REFD(N_{k-1}) then by (15) $p(s_k|m^k)$ is a REFD(max$\{2M, N_{k-1}\}$) , so if s_1 has a REFD(N) given by (16) then $p(s_k|m^k)$ is a REFD(Max$\{2M,N\}$) for any k , with a finite number of recursive formulae existing for the coefficients.

These formulas will not be enumerated here since they are tedious to list for an arbitrary integer M.

We exemplify this procedure for $p = 3$ where the vector-function h has a very simple form so that the computations are not unduly cumbersome:

$$h(s_k) = \begin{bmatrix} \cos\theta_k \\ \cos\phi_k \sin\theta_k \\ \cos\psi_k \sin\theta_k \end{bmatrix}$$

the components being three of the direction cosines of the matrix representation of s_k of the form (1).

Let us assume that $\{v_k\}$ is a vector Gaussian process with each term v_k having zero mean and 3×3 covariance matrix R_k, and that s_0 has the REFD(N) (16). Now

$$p(m_k \mid s_k) = (2\pi)^{-3/2}(\det R_k)^{-1/2}\exp - \tfrac{1}{2}[m_k - h(s_k)]' R_k[m_k - h(s_k)] .$$

Using (2), (3), and (8) in the following calculations, we first write $h(s_k)$ in the form

$$h(s_k) = \begin{bmatrix} D^1(s_k)_{00} \\ \frac{i}{\sqrt{2}}[D^1(s_k)_{10} + D^1(s_k)_{-1,0}] \\ \frac{i}{\sqrt{2}}[D^1(s_k)_{01} + D^1(s_k)_{0,-1}] \end{bmatrix}$$

so that

$$[m_k-h(s_k)]'R_k[m_k-h(s_k)] = [m_k^1-D^1(s_k)_{00}]^2R_k^{11}+2[m_k^1-D^1(s_k)_{00}][m_k^2-\frac{i}{\sqrt{2}}(D^1(s_k)_{10}$$

$$+ D^1(s_k)_{-1,0})]R_k^{12}+2[m_k^1-D^1(s_k)_{00}][m_k^3-\frac{i}{\sqrt{2}}(D^1(s_k)_{01}+D^1(s_k)_{0,-1}]R_k^{13}$$

$$+ [m_k^2-\frac{i}{\sqrt{2}}(D^1(s_k)_{10}+D^1(s_k)_{-1,0})]^2R_k^{22}+2[m_k^2-\frac{i}{\sqrt{2}}(D^1(s_k)_{10}+D^1(s_k)_{-1,0})]$$

$$\times [m_k^3-\frac{i}{\sqrt{2}}(D^1(s_k)_{01}+D^1(s_k)_{0,-1}]R_k^{23}+[m_k^3-\frac{i}{\sqrt{2}}(D^1(s_k)_{01}+D^1(s_k)_{0,-1})]^2R_k^{33}$$

$$= \sum_{i,j=1}^{3} R_k^{ij}m_k^i m_k^j - 2\sum_{j=1}^{3} m_k^j R_k^{1j}D^1(s_k)_{00} - \sqrt{2}\, i \sum_{j=1}^{3} m_k^j R_k^{j2}D^1(s_k)_{10}$$

$$- \sqrt{2}\, i \sum_{j=1}^{3} m_k^j R_k^{j2}D^1(s_k)_{-1,0} -\sqrt{2}\, i \sum_{j=1}^{3} m_k^j R_k^{j3}D^1(s_k)_{01} - \sqrt{2}\, i \sum_{j=1}^{3} m_k^j R_k^{j3}D^1(s_k)_{0,-1}$$

$$+ \frac{1}{3} R_k^{11}[D^0(s_k)_{00}+2D^2(s_k)_{00}]+\sqrt{\frac{2}{3}}\, i\, R_k^{12}[D^2(s_k)_{10}+D^2(s_k)_{-1,0}]$$

$$+\sqrt{\frac{2}{3}}\, i\, R_k^{13}[D^2(s_k)_{0,-1}+D^2(s_k)_{01}]-R_k^{22}[\frac{1}{\sqrt{6}} D^2(s_k)_{20}+ \frac{1}{3}[D^2(s_k)_{00}$$

$$- D^0(s_k)_{00}]+ \frac{1}{\sqrt{6}} D^2(s_k)_{-2,0}]- \frac{1}{2} R_k^{23}[D^2(s_k)_{11}-D^1(s_k)_{11}+D^1(s_k)_{1,-1}$$

$$+D^2(s_k)_{1,-1}+D^1(s_k)_{-1,1}+D^2(s_k)_{-1,1}+D^2(s_k)_{-1,-1}-D^1(s_k)_{-1,-1}]$$

$$- R_k^{33}[\frac{1}{\sqrt{6}}D^2(s_k)_{02}+ \frac{1}{3} (D^2(s_k)_{00}-D^0(s_k)_{00})+ \frac{1}{\sqrt{6}} D^2(s_k)_{0,-2}].$$

Assuming that

$$p(s_{k-1}|m^{k-1}) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell,k-1} D^{\ell}(s_{k-1})_{mn}$$

and using (15) and (19) we obtain

$$p(s_k | m^k) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell k} D^{\ell}(s_k)_{mn}$$

where a_{00}^{0k} is a normalizing constant,

$$a_{00}^{1k} = -2 \sum_{j=1}^{3} m_k^j R_k^{1j} + \sum_{j=-1}^{1} a_{j0}^{1,k-1} D^1(w_{k-1}^{-1})_{j0} ,$$

$$a_{mn}^{1k} = -\sqrt{2}\, i \sum_{j=1}^{3} m_k^j R_k^{jn} + \sum_{j=-1}^{1} a_{jn}^{1,k-1} D^1(w_{k-1}^{-1})_{jm} , \quad |m+n| = 1, \quad mn = 0 ;$$

$$a_{mn}^{1k} = -\frac{1}{2} R_k^{23} mn + \sum_{j=-1}^{1} a_{jn}^{1,k-1} D^1(w_{k-1}^{-1})_{jm} , \quad |mn| = 1$$

(25) $$a_{00}^{2k} = \frac{1}{3}(2R_k^{11} - R_k^{22} - R_k^{33}) + \sum_{j=-2}^{2} a_{j0}^{2,k-1} D^2(w_{k-1}^{-1})_{j0}$$

$$a_{mn}^{2k} = -\frac{1}{\sqrt{6}} R_k^{pp} + \sum_{j=-2}^{2} a_{jn}^{2,k-1} D^2(w_{k-1}^{-1})_{jm} , \quad mn = 0 , \quad |m+n| = 2 ,$$

$$p = 2^{\delta(|m|,2)} \times 3^{\delta(|n|,2)} ;$$

$$a_{mn}^{2k} = \sqrt{\frac{2}{3}}\, i\, R_k^{1p} + a_{mn}^{2,k-1} + \sum_{j=-2}^{2} a_{jn}^{2,k-1} D^2(w_{k-1}^{-1})_{jm} , \quad mn = 0 ,$$

$$|m+n| = 1 , \quad p = 2^{\delta(|m|,1)} \times 3^{\delta(|n|,1)} ;$$

$$a_{mn}^{2k} = -\frac{1}{2} R_k^{33} + \sum_{j=-2}^{2} a_{jn}^{2,k-1} D^2(w_{k-1}^{-1})_{jm} , \quad |mn| = 1 , \quad |m| + |n| = 2$$

$$a_{mn}^{\ell k} = \sum_{j=-\ell}^{\ell} a_{jn}^{\ell k} D^{\ell}(w_{k-1}^{-1})_{jm} , \quad \text{otherwise.}$$

[We have used the symbol $\delta(a,b)$ to denote the number 1 when $a = b$ and 0 otherwise]. Now optimal estimates can be obtained exactly as in the last section.

VI. Detection for Rotational Processes

In this section we consider the detection problem of determining the presence or absence of a rotational signal process using each of the measurement processes given in the past two sections. It will be shown that the likelihood ratio of the presence to the absence of the signal can be efficiently ascertained when REFD's are utilized.

Let us first refer to the rotational signal and measurement processes $\{s_k\}$ and $\{m_k\}$ described by (13) and (14) where s_0 and $\{v_k\}$ have REFD's given by (16) and (17) which are independent processes. We now consider the following hypotheses that the signal rotation s_k is present and absent respectively:

$$H_1 : \quad m_k = v_k \circ s_k$$

$$H_0 : m_k = v_k$$

The likelihood ratio that we want to compute is given by

$$L(m^k) \triangleq \frac{p(m^k | H_1)}{p(m^k | H_0)}$$

Since $\{s_k\}$ and $\{v_k\}$ are independent, we have

$$\begin{aligned} p(m^k | H_1) &= E[p(m^k | H_1, s^k) | H_1] \\ &= E[\prod_{j=1}^{k} \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell j} D^{\ell}(m_j \circ s_j^{-1})_{mn}] \end{aligned}$$

while

$$p(m^k | H_0) = \prod_{j=1}^{k} \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell j} D^{\ell}(m_j)_{mn}$$

so that

$$L(m^k) = E[\exp \sum_{j=1}^{k} \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell j}[D^\ell(m_j \circ s_j^{-1})_{mn} - D^\ell(m_j)_{mn}]$$

$$= E[\exp \sum_{j=1}^{k} W(s_j,j,m_j)]$$

where

$$W(s_j,j,m_j) = \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell j}[D^\ell(m_j \circ s_j^{-1})_{mn} - D^\ell(m_j)_{mn}] \tag{26}$$

We now state a lemma whose proof is identical to that of a corresponding lemma proved in [1, pp.14-15] with the circular function replaced with the functions $D^\ell(s)_{mn}$:

<u>Lemma 1.</u> Let $\{s_k\}$ be a random signal process on SO(3) and $\{v_k\}$ a white noise process on SO(3) which is independent of $\{s_k\}$, having the rotational exponential Fourier density (23). Let

$$T = \{t_1,\ldots,t_q\}$$

$$T' = \{t'_1,\ldots,t'_{q'}\}$$

$$S_T = \{s_{t_1},\ldots,s_{t_q}\}$$

where q,q',t_i, and t'_i are positive integers. If the measurement process is described by $m_k = v_k \circ s_k$ then the conditional joint probability density is

$$p(S_T|m^k,S_{T'}) = \frac{E[\exp H(1,k)|S_T,S_{T'}]p(S_T|S_{T'})}{E[\exp H(1,k)|S_{T'}]}$$

where $H(1,k) = \sum_{j=1}^{k} W(s_j,j,m_j)$ and $W(s_j,j,m_j)$ is given by (26).

We can now obtain a recursive formula for $L(m^k)$ by using the smoothing property of a conditional expectation and Lemma 1:

$$L(m^k) = \int_\Sigma E[\exp \sum_{j=1}^{k} W(s_j,j,m_j) \mid s_k]p(s_k)ds_k$$

$$= \int_\Sigma E[\exp \sum_{j=1}^{k-1} W(s_j,j,m_j) \mid s_k]\exp W(s_k,k,m_k)p(s_k)ds_k$$

$$= \int_\Sigma E[\exp \sum_{j=1}^{k-1} W(s_j,j,m_j)]p(s_k \mid m^{k-1})\exp W(s_k,k,m_k)ds_k$$

$$= E[\exp H(1,k-1)] \int_\Sigma p(s_k \mid m^{k-1} \exp W(s_k,k,m^k)ds_k$$

$$= L(m^{k-1})Q(m^k)$$

where $\int_\Sigma f(s_k)ds_k \triangleq \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(\phi_k,\theta_k,\psi_k)\sin \theta_k d\phi_k d\theta_k d\psi_k$

and $Q(m^k) \triangleq \int_\Sigma p(s_k \mid m^{k-1})\exp W(s_k,k,m_k)ds_k$.

The integral $Q(m^k)$ can be evaluated in the following form which is obtained with the assistance of (19), (26) and (6):

$$Q(m^k) = \int_\Sigma \exp[W(s_k,k,m_k) + \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell,k-1} D^\ell(w_{k-1}^{-1} \circ s_k)_{mn}]ds_k$$

$$= \int_\Sigma \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} \sum_{j=-\ell}^{\ell} \{a_{jn}^{\ell,k-1} D^\ell(w_{k-1}^{-1})_{jm}D^\ell(s_k)_{mn}$$

$$+ b_{mn}^{\ell k}[D^\ell(m_k)_{mj}D^\ell(s_k^{-1})_{jn} - D^\ell(m_k)_{mn}]\}ds_k$$

$$= \exp\{ - \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} b_{mn}^{\ell k} D^{\ell}(m_k)_{mn}\} \cdot \int_{\Sigma} \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} C_{mn}^{\ell k} D^{\ell}(s_k)_{mn} ds_k$$

where $$C_{mn}^{\ell k} = \sum_{j=-\ell}^{\ell} \{a_{jn}^{\ell k} D^{\ell}(w_{k-1}^{-1})_{jn} + (-1)^{m+n} b_{j,-m}^{\ell k} D^{\ell}(m_k)_{j,-n}\}$$

Let us now study the detection problem that results from using the signal process discussed in Section V described by (21) and (22). Suppose we have the hypotheses

$$H_1 : m_k = h(s_k) + v_k$$

$$H_0 : m_k = v_k$$

Using (23) it is found that the likelihood ratio is

$$L(m^k) = E[\exp H(1,k)]$$

$$H(1,k) = \sum_{j=1}^{k} W(s_j, j, m_j)$$

$$W(s_j,j,m_j) = \sum_{\ell=0}^{M} \sum_{m,n=-\ell}^{\ell} C_{mn}^{\ell} D^{\ell}(s_j)_{mn} + \sum_{\ell,\ell'=0}^{M} \sum_{m,n=0}^{\ell} \sum_{m',n'=0}^{\ell'} C^{\ell\ell'}(m,n,m',n')$$

$$\times D^{\ell}(s_j)_{mn} D^{\ell'}(s_j)_{m'n'}$$

$$C_{mn}^{\ell} = \frac{1}{2} \sum_{i,j=1}^{p} R_k^{ij} [m_k^j h_{mn}^{\ell j} + m_k^i h_{mn}^{\ell i}]$$

$$C^{\ell\ell'}(m,n,m',n') = -\frac{1}{2} \sum_{i,j=1}^{p} R_k^{ij} h_{mn}^{\ell i} h_{m'n'}^{\ell' j}$$

An argument identical to that used in the previous detection problem yields

$$L(m^k) = \int_\Sigma E[\exp H(1,k) | s_k] p(s_k) ds_k$$

$$= L(m^{k-1}) \int_\Sigma p(s_k | m^{k-1}) \exp W(s_k, k, m_k) ds_k$$

$$= L(m^{k-1}) Q(m^k)$$

Since it has been shown in the last section that the conditional density $p(s_k | m^{k-1})$ is a rotational exponential Fourier density of the form

$$\exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell k} D^\ell(s_k)_{mn}$$

where the coefficients can be computed recursively, we can write the integral $Q(m^k)$ in the form

$$Q(m^k) = \int_\Sigma \exp \sum_{\ell=0}^{M} \sum_{m,n=-\ell}^{\ell} C_{mn}^\ell D^\ell(s_k)_{mn}$$

$$+ \sum_{\ell,\ell'=0}^{M} \sum_{m,n=-\ell}^{\ell} \sum_{m',n'=-\ell}^{\ell'} C^{\ell\ell'}(m,n,m',n') D^\ell(s_k)_{mn} D^{\ell'}(s_k)_{m'n'}$$

$$+ \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell k} D^\ell(s_k)_{mn} \quad ds_k .$$

Consequently, $Q(m^k)$ is of the form

$$\int_\Sigma \exp \sum_{\ell=0}^{P} \sum_{m,n=-\ell}^{\ell} \alpha_{mn}^{\ell k} D^\ell(s_k)_{mn} ds_k$$

where $P = \max\{N, 2M\}$ and the coefficients $\alpha_{mn}^{\ell k}$ which are functions of $a_{mn}^{\ell k}$, C_{mn}^ℓ, $h_{mn}^{\ell i}$, R_k, and w_k are obtained using (24). Hence the computational scheme for the likelihood ratio $L(m^k)$ is finite dimensional. Rather than attempting to exhibit the formulas in the general case we will produce them for the simple example at the end of Section V where

$$h(s_k) = \begin{bmatrix} \cos \theta_k \\ \cos \phi_k \sin \theta_k \\ \cos \psi_k \sin \Theta_k \end{bmatrix}$$

and $\{v_k\}$ is the Gaussian with zero mean and covariance matrix R_k.

Since it was shown that the conditional density $p(s_k|m^k)$ can be written as

$$p(s_k|m^k) = \exp \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell k} D^{\ell}(s_k)_{mn}$$

with the coefficients given by (25), it is easily verified that the likelihood ratio is

$$L(m^k) = E[\exp H(1,k)]$$

$$H(1,k) = \sum_{j=1}^{k} W(s_j, j, m_j)$$

$$W(s_j, j, m_j) = \sum_{\ell=0}^{N} \sum_{m,n=-\ell}^{\ell} a_{mn}^{\ell j} D^{\ell}(s_j)_{mn} - \frac{1}{2} \sum_{s,t=1}^{3} R_j^{st} m_j^s m_j^t$$

and

$$Q(m^k) = \exp\{a_{00}^{0k} + a_{00}^{0,k-1} - \frac{1}{2} \sum_{s,t=1}^{3} R_k^{st} m_k^s m_k^t\}$$

$$\times \int_{\Sigma} \exp \sum_{\ell=1}^{N} \sum_{m,n=-\ell}^{\ell} C_{mn}^{\ell k} D^{\ell}(s_k)_{mn} ds_k$$

$$C_{mn}^{\ell k} = a_{mn}^{\ell k} + \sum_{j=-\ell}^{\ell} a_{jn}^{\ell,k-1} D^{\ell}(w_{k-1})_{jm}$$

VII. Conclusions

In this paper we have formulated and solved estimation and detection problems for discrete-time processes that arise on SO(3). The signal process s_{k+1} is assumed to be obtained from s_k by a concatenation which is the successive rotations s_k and w_k where the latter is a known rotation, while two types of measurement processes have been used: (i) the observation m_k is the concatenation of s_k and a noise rotation v_k, and (ii) the observation m_k is a smooth function of s_k corrupted by additive white noise.

An error criterion which differs from a least squares criterion such as appears in [8]and [9] has been presented together with the resulting optimal estimates. In addition a new density function, the rotational exponential Fourier density has been introduced which can be used to approximate probability densities on SO(3) that are continuous.

Since this class of density functions is closed under the operation of taking conditional densities, we have been able to obtain recursive schemes for optimal estimation and detection for a rather large class of problems. However recursive schemes for the analogous problem when the driving term w_k is a stochastic process have not been resolved since these densities do not have the property of being closed under convolution. Perhaps a different representation for densities on SO(3) or a different model of the signal process can be found that will yield a recursive solution to this problem.

It is believed that the procedures of this paper can be extended to SO(n) for $n > 3$ by using the appropriate special functions defined on SO(n) that are analogous to the functions D^{ℓ}_{mn} on SO(3).

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

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18. SUPPLEMENTARY NOTES

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Exponential Fourier density, Optimal estimation, Detection, Additive noise, Conditional distribution, Rotational noise, Rotational harmonics, 3-dimensional rotation group, Recursive schemes, Error Criterion.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, we will present a new representation of a probability density function on the three dimensional rotation group, SO(3), which generalizes the exponential Fourier densities on the circle. As in the circle case, this class of densities on SO(3) is also closed under the operation of taking conditional distributions. Several simple multistage estimation and detection models will be considered in this paper. The closure property enables us to determine the sequential conditional (Continued)

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20. (Continued)

densities by recursively updating a finite and fixed number of coefficients. It also enables us to express the likelihood ratio for signal detection explicitly in terms of the conditional densities.

An error criterion, which is compatible with a Riemannian metric, will be introduced and discussed in this paper. The optimal orientation estimates with respect to this error criterion will be derived for a given probability distribution, illustrating how the updated conditional densities can be used to sequentially determine the optimal estimates on SO(3).